Ph.D. Qualification Examination

Algorithms (Apr. 2007)

(1) (15%) Give asymptotically tight upper bounds for $T(n)$ in each of the following recurrences.
  (a) $T(n) = 4T(n/2) + n$, $T(1) = 1$.
  (b) $T(n) = 4T(n/2) + n^2$, $T(1) = 1$.
  (c) $T(n) = 4T(n/2) + n^3$, $T(1) = 1$.

(2) (15%) Let $A[1 \ldots n]$ be an array of n distinct numbers. A pair $(A[i], A[j])$ is said to be an *inversion* if $i < j$ but $A[i] > A[j]$. Design an $O(n \log n)$ algorithm for counting the number of inversions.

(3) (10%) Prove the equality

$$\lfloor \log_2 n \rfloor + 1 = \lceil \log_2 (n+1) \rceil \text{ for } n \geq 1.$$

(4) (10%) A certain problem can be solved by an algorithm whose running time is in $O(n^{\log_2 n})$. Which of the following assertions is true?
  (a) The problem is tractable.
  (b) The problem is intractable.
  (c) None of the above.

(5) (10%) There are $2n$ glasses standing next to each other in a row, the first n of them filled with a soda drink, while the remaining n glasses are empty. What is the minimum number of glass moves by making the classes alternate in a filled-empty-filled-empty pattern? Note that a move is an exchange of two glasses.

(6) (20%) Design an algorithm for the *change-making problem*: given an amount n and unlimited quantities of coins of each of the denominations $d_1, d_2, \ldots, d_m$, where $d_i > d_{i+1}$ for $1 \leq i \leq m-1$, find the smallest number of coins that add up to n or indicate that the problem does not have a solution.

(7) A graph is *bipartite* if all its vertices can be partitioned into two disjoint subsets X and Y such that every edge connects a vertex in X with a vertex in Y.
  (a) (10%) Design a DFS-based algorithm for checking whether a graph is bipartite.
  (b) (10%) Design a BFS-based algorithm for checking whether a graph is bipartite.