1. [15%] Define $H_k = 1+\frac{1}{2}+...+\frac{1}{k}$ *for* $k \geq 1$. Please use mathematical induction to prove that $H_{2^n} \geq 1+\frac{n}{2}$ *for* $n \geq 0$.

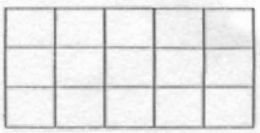
2. [10%] Given $f = \{(a,b),(b,a),(c,b)\}$, a function from $X = \{a,b,c\}$ to X:

(a) Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.

(b) Define $f^n = f \circ f \circ ... \circ f$ to be the n-fold composition of f with itself. Find f^0 and f^{623}.

3. [20%]

(a) How many routes are there from the lower-left corner to the upper-right corner of the following 3×5 grid in which we are restricted to traveling only to the right or upward?



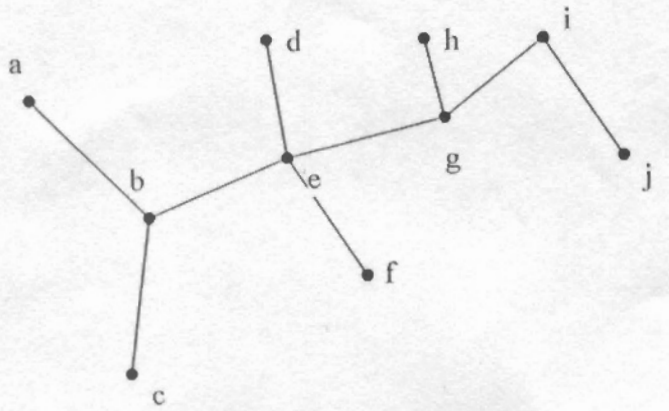
(b) Divide the routes into classes based on when the route first meets the top edge to derive the formula $\sum_{k=0}^{n} C(k+m-1,k) = C(m+n,m)$.

4. [15%] The enzyme-linked immunosorbent assay(ELISA) test is used to detect antibodies in blood and can indicate the presence of the HIV virus. Approximately 15 percent of the patients at one clinic have the HIV virus. Furthermore, among those that have the HIV virus, approximately 95 percent test positive on the ELISA test. Among those that do not have the HIV virus, approximately 2 percent test positive on the ELISA test. Find the probability that a patient has the HIV virus if the ELISA test is positive.

5. [10%] Suppose that algorithm A requires $\lceil n \log_2 n \rceil$ comparisons to sort n items and algorithm B requires $\lceil n^2/4 \rceil$ comparisons to sort n items. For which n is algorithm B superior to algorithm A?

6. [15%] Draw the tree T of the following figure as a rooted tree with b as root. What is

the height of the resulting tree?



7. [15%] Prove $n(1+x)^{n-1} = \sum_{k=1}^{n} C(n,k)kx^{k-1}$.