Discrete Mathematics and Linear Algebra

1. (10%) Show that if the sum of $n+1$ positive integers is $2n$, then for any positive integer k less than $2n$, there is a subset of these $n+1$ integers whose sum is k.

2. (10%) Find the conjunctive normal form of $f(x,y,z) = xy + \overline{y}z$.

3. (10%) Show that a directed graph has an Euler cycle if and only if it is strongly connected and the indegree of every vertex is equal to its outdegree.

4. How many permutations of the numbers 1, 2, 3, 4, 5, 6, and 7 are there if
   (a) (5%) there must be two or three numbers between 1 and 2?
   (b) (5%) there must not be two or three numbers between 1 and 2?
   (c) (5%) the first four numbers must be chosen from the numbers 1, 3, 5, and 7?
   (d) (5%) the numbers 1, 3, 5, and 7 must be together?

5. (15%) Find an orthonormal basis for the vector space generated by the vectors $(1,1,0,1)$, $(1,-2,0,0)$, and $(1,0,-1,2)$.

6. (15%) Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

7. Let $u = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, and $z = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$. Answer the following questions and explain your answer.
   (a) (5%) Are the sets $\{u,v\}$, $\{u,w\}$, $\{u,z\}$, $\{v,w\}$, $\{v,z\}$, and $\{w,z\}$ each linearly independent?
   (b) (5%) Does the answer to (a) imply that $\{u,v,w,z\}$ is linear independent?
   (c) (5%) Can we determine if $\{u,v,w,z\}$ is linearly independent by checking if w is a linear combination of u, v, and z?
   (d) (5%) Is $\{u,v,w,z\}$ linearly dependent?