Discrete Mathematics

1. (20%) The **eccentricity** of a vertex v in a tree is the length of the longest simple path from v. The **center** of a tree is a vertex with the least eccentricity. Show that there are at most two centers for any tree.

2. Determine if each relation from $\{a, b, c, d\}$ to $\{0, 1, 2, 3, 4\}$ is a function.

   (a) (5%) $\{(a, 0), (b, 1), (c, 1), (d, 3)\}$

   (b) (5%) $\{(a, 2), (b, 2), (b, 4), (c, 1), (d, 3)\}$

   (c) (5%) $\{(a, 1), (b, 1), (c, 1), (d, 1)\}$

   (d) (5%) $\{(a, 0), (b, 1), (c, 2)\}$

3. Solve the following recurrence relations.

   (a) (10%) $a_1 = 1$, $a_n = a_{n-1} + n^2$, $n \geq 2$

   (b) (10%) $b_1 = 1$, $b_n = 2b_{n-1} + (2^n - 1)$, $n \geq 2$

4. (20%) Let N be the set of positive integers. Show that the power set of N is not countably infinite.

5. (20%) Let $x > -1$ be a real number. Prove that $(1 + x)^n \geq 1 + nx$ for all $n \geq 0$.