

## 本考科禁用計算機

(請於此線以下開始出題)

1. [10%] Let  $H_n$  be the  $n$ th harmonic number  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ , show that  $H_n$  is  $O(\log n)$ .
2. [15%] Show that  $\sqrt{2}$  is irrational by giving a proof by contradiction.
3. [15%] An explorer discovers an island during his travel. There are two types of residents- those who always tell the truth and those who always lie. He wants to determine whether a particular resident always lies or always tells the truth by asking one question.
  - (a) Explain why the question "Are you a liar?" does not work. [5%]
  - (b) Find a question that the explorer can use to determine whether the resident always lies or always tells the truth. Explain your answer. [10%]
4. [10%] Show that if  $2^n - 1$  is prime, then  $n$  is prime.
5. [10%] Determine the number of vectors  $(x_1, x_2, \dots, x_n)$ , such that each  $x_i$  is either 0 or 1 and  $x_1 + x_2 + \dots + x_n \geq k$ .
6. [10%] Delegates from 10 countries, including Russia, France, England, and the United States, are to be seated in a row. How many different seating arrangements are possible if the French and English delegates are to be seated next to each other, and the Russian and U.S. delegates are not to be next to each other?
7. [10%] Solve the recurrence  $C_N = 2C_{N/2} + N$ , for  $N \geq 2$  with  $C_1 = 0$ .
8. [10%] Proof that: if a set  $S$  is countable infinite, the  $S \times S$  is countable infinite.
9. [10%] Find the simplest expression for the following events
  - (a) (3%)  $(E \cup F) \cap (E \cup F')$
  - (b) (3%)  $(E \cup F) \cap (E' \cup F) \cap (E \cup F')$
  - (c) (4%)  $(E \cup F) \cap (F \cup G)$